Project 1

Diffusion Equation

Due: Mon., Oct. 18, 2010 at 6:00 pm

Two parallel plates extended to infinity are a distance of $h = 4\,cm$ apart. The fluid within the plates has a kinematic viscosity of $\nu = 0.000217\,m^2/s$ and density of $800\,kg/m^3$. The upper plate is stationary and the lower one is suddenly set in motion with a constant velocity of $U_0 = 40\,m/s$. The governing equation is Navier-Stokes simplified as:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

where y is the cross-stream direction and $u(y,t)$ is the streamwise velocity component. Use a first-order forward-time and second-order central space (FTCS) scheme to discretize the PDE. The analytical solution for this PDE is given by:

$$u(\eta) = U_0 \left(\sum_{n=0}^{\infty} erfc(2n\eta_1 + \eta) - \sum_{n=1}^{\infty} erfc(2n\eta_1 - \eta) \right)$$

where $\eta = y/2\sqrt{\nu t}$, $\eta_1 = h/2\sqrt{\nu t}$ and $erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-r^2}\,dr$ is the complementary error function.

1. Derive the truncation error of the finite-difference equation. Is FTCS scheme consistent?
2. The stability condition for FTCS is given by: $d = \nu \Delta t / \Delta y^2 \leq 0.5$. Verify this condition numerically by examining the velocity profile for different Δt and Δy.
3. Decrease Δy (adjust Δt accordingly to satisfy the stability condition). At what Δy value does the solution become independent of Δy. This is called the *grid independent* solution.
4. Show that the accuracy of the solution improves on a finer mesh (as Δy decreases). The error can be calculated as:

$$\%error = \frac{Analytical - Numerical}{Analytical} \times 100$$

5. Plot the velocity profile for $t = 0, 0.18, 1.08\,sec$ and compare the numerical results with the analytical solution.
6. Continue the numerical solution in time untill it reaches steady-state. Compare this solution with the analytical solution as well as that obtained from the PDE directly (*i.e.* by solving the $\frac{\partial^2 u}{\partial y^2} = 0$ analytically).